The Open University

MS221
Exploring Mathematics

Chapter D4

Proof and reasoning

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3.1

Contents

Study guide

This chapter contains three sections, each of which comprises a single study session. Section 1 is based on an audio band; otherwise the chapter involves only the printed text. The study pattern which we recommend is as follows.

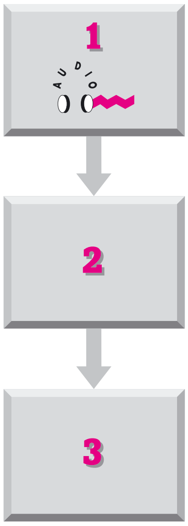
Study session 1: Section 1.

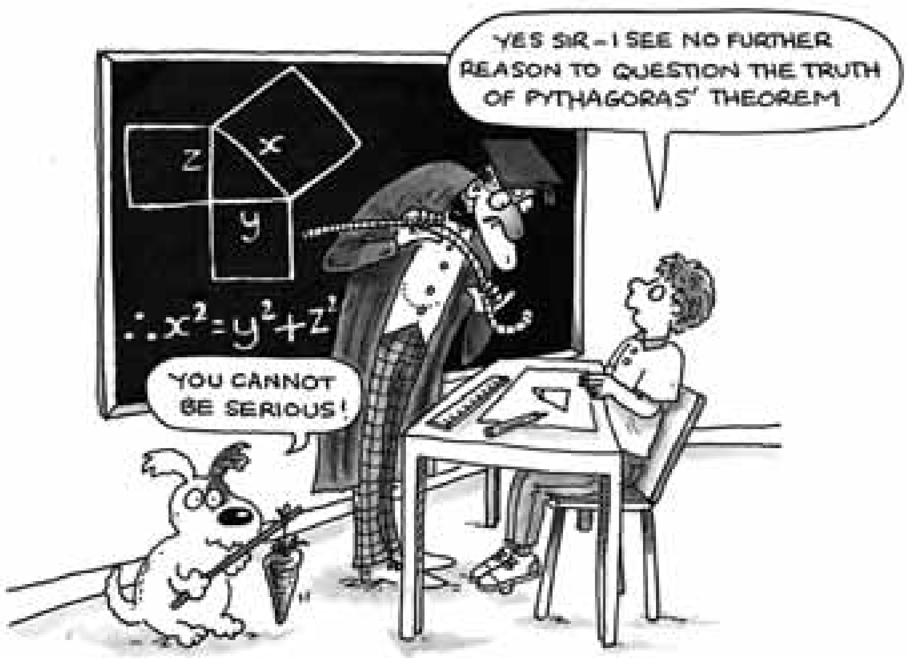
Study session 2: Section 2.

Study session 3: Section 3.

However, it would be possible to proceed with your study of Sections 2 and 3 before listening to the audio band in Section 1 if you find it essential to do so.

> The optional Video Band D(iv), *Algebra workout – Mathematical induction*, could be viewed at any stage during your study of this chapter.





SOME ARGUMENTS ARE MORE CONVINCING THAN OTHERS

Introduction

The subject of proof has been a theme throughout MS221, and our aim here is to review and consolidate ideas about this. We begin the chapter by looking at several examples of proofs, some from earlier in the course and some new. These examples are chosen to illustrate commonly occurring types of proof. Recognition of types of structures that are often used should help you both in constructing proofs for yourself and in studying texts which contain proofs. We next introduce a few pieces of notation that help us to express in a clear and succinct way the structure of proofs.

As well as considering types of proof that have been used earlier, we shall introduce one new form of proof. This is proof by *mathematical induction*, which is particularly valuable in proving results to be true for all natural numbers. Such a result is $\displaystyle\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$, for $n = 1, 2, 3, \ldots$, giving a formula for the sums of cubes.

Different disciplines have their own approaches to establishing meaning and truth, and in many cases knowledge is built up by the interpretation of evidence. Such evidence may already exist in the forms of fossils, say, observed phenomena, or historical documents, or may be intentionally generated through carefully designed experiments. Much of mathematics derives, in the first instance, from the observation of real phenomena, and a wish to explain such phenomena. But mathematics differs from science in that it does not refer back to practical experience or experiment in order to justify its results. In contemporary mathematics, it is typical to present knowledge in an orderly way, starting with a clear statement of what will be assumed about the subject being discussed. For example, in Chapter D3 we described the axioms defining 'a group', and then derived information about groups by logical deduction based on these axioms, with all terms used in the process being precisely defined.

This approach to the organisation of mathematical knowledge is relatively modern. That is not to suggest that earlier mathematicians did not seek to justify their results. But at times when new mathematical techniques were being invented to model observed phenomena, the derivation of accurate predictions provided a strong indication that the techniques being used were valid. Techniques that appear to work are hard to ignore, even if they involve methods that are not fully understood.

Even today, there is not total agreement about what constitutes acceptable proof, or as to precisely what axioms should be chosen on which to base the fundamental ideas of mathematics, such as that of 'a set'. These considerations relate to the underlying foundations and philosophy of mathematics, though. For nearly all mathematics, there *are* generally accepted standards of mathematical proof that are clear enough, and it is with these that we are concerned here.

1 Aspects of proof

Note that in mathematics a statement must be either true or false (but not both).

This course has included many results, sometimes with their proofs. A *result* is a mathematical statement, usually one of some significance, that is known to be true. More informal words for result are: fact or claim. Sometimes an important result is given a more formal name such as: theorem, lemma, or corollary. On the other hand, *proof* is often referred to as reasoning, argument, justification or deduction. We invite you to start your work on proof by recalling styles of reasoning that have been used throughout the course.

Activity 1.1 Types of reasoning

Some examples of results from earlier in the course are given below. Look at each of these results and try to recall what type of proof was used.

See Chapter A1, Section 2.

(a) The number $\phi = \frac{1}{2}(1+\sqrt{5})$ is irrational.

See Chapter A1, Section 3.

(b) The nth Fibonacci number F_n is given by the formula

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \psi^n),$$

where $\phi = \frac{1}{2}(1+\sqrt{5})$ and $\psi = \frac{1}{2}(1-\sqrt{5})$.

See Chapter B1, Section 1.

(c) Suppose that the sequence x_n is generated by iteration of the real function f, and that x_n converges to the limit l. If f is continuous, then l is a fixed point of f.

See Chapter B1, Section 5.

(d) For $x \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + x^n.$$

See Chapter C1, Section 1.

(e) The derived function of $f(x) = x^4$ is $f'(x) = 4x^3$.

See Chapter D2, Section 2.

(f) A positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

Comment

(a) The fact that $\phi = \frac{1}{2}(1+\sqrt{5})$ is irrational was established by geometric reasoning, based on properties of a regular pentagon.

(b) The formula for F_n was obtained as a special case of a formula for the general solution of a linear second-order recurrence system.

(c) The proof used the assertion that if x_n tends to l, then $f(x_n)$ tends to $f(l)$, which appears plausible in view of the *informal* definition of a 'continuous function'.

(d) The proof of this binomial expansion was based on an argument that the pattern of the expansion for small values of n remains the same as n increases.

(e) This formula was deduced from the definition of 'derivative', but that definition was based on an *informal* notion of the 'limit of a function'.

(f) This result was proved carefully using properties of congruences.

Several different types of proof were mentioned in the Comment on Activity 1.1, including geometric and algebraic reasoning, and generalisation of a pattern. Geometric reasoning, based on diagrams, can be very convincing but may not be entirely rigorous (for example, your diagram may only work in some circumstances). Algebraic reasoning is often considered to be more rigorous, but again it is possible for an algebraic proof to be invalid in some cases (for example, when the variables involved take 'awkward' values). Also, in an algebraic proof it can be difficult to 'see' what is happening. Proof by generalisation of pattern may be convincing but it is not valid unless an argument is given to show why the pattern continues.

Whatever type of proof is used, it should ideally have the following characteristics.

- ◇ All assumptions are stated.
- ◇ All terms used are defined.
- ◇ There is a train of argument showing why the conclusion follows from the assumptions.
- ◇ All cases are covered.

When reading a proof in text, you may sometimes find that the proof given is clear and sufficient, and can be read and understood as it stands. On other occasions, your understanding of a proof may be quite limited. In such cases, it can be useful to try out some examples. Such 'specialising' helps to establish the nature of the result being proved; this may turn out to be an important precursor to understanding the proof. Also, specialisation may (but may not!) reveal the underlying reason why the result holds in general.

The audio band that follows concentrates on 'reading' proofs. It examines common structures (overall approaches) used in proofs. The most natural approach when presenting a proof is to start with known information, and to work forward in a sequence of deductions, to reach the desired conclusion. Not all proofs are constructed like this, though. For example, in 'proof by contradiction', we start by assuming that the result to be proved is not true, and argue that this assumption (together with other known information) must lead to a contradiction. When reading a proof, it is useful to recognise which overall proof structure is being used.

There is no set technique for constructing proofs, and consequently this task can be a difficult one! Before you start it is important to be clear as to the result that is to be proved. Note what you *know* to be true, and what you *want* to prove. Look for a line of reasoning, leading from 'what you know' to 'what you want'. Although you should aim to end up presenting a proof that looks neat and compact, you are unlikely to arrive at it by a tidy process. Expect to go down blind alleys, and to work from both ends (both from 'what you know' and from 'what you want') in the hope that you can make things meet in the middle.

Before listening to the audio band, we ask you to think about a couple of situations where a proof is required.

Activity 1.2 Specialising, conjecturing and convincing

Recall that a *conjecture* is a statement that we feel may be true, but of which we have no proof.

Spend a few minutes (at most) working on the Two Sums Problem, stated in the box below. When you think you have a conjecture as to what is happening, try to convince yourself that your conjecture is true. Then try to express your argument clearly enough to convince someone else.

Two Sums Problem

- ◇ Take any two real numbers whose sum is 1.
- ◇ Square the larger and add the smaller.
- ◇ Then square the smaller and add the larger.

Which answer will be bigger?

We suggest that you specialise by looking at some particular cases. We shall revisit the problem at the end of the section.

Activity 1.3 Thinking about a proof

Increasing functions were introduced in Chapter B1, Section 1.

Below is a result about increasing functions. You are not asked to prove the result at this stage. For now, identify

- ◇ what you know, and
- ◇ what you want to prove.

Result

Let f and g be increasing real functions with domain $\mathbb{R}$. Let h be the sum of these two functions; that is,

$$h(x) = f(x) + g(x) \quad (x \in \mathbb{R}).$$

Then h is also an increasing function.

This result is discussed in Frames 14–16 associated with the audio band. Frame 14 shows all that you are expected to do at present.

AUDIO

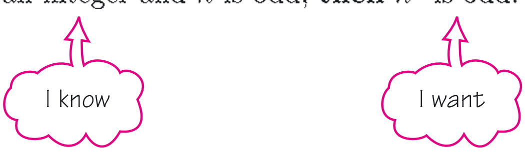
Now listen to CDA5494 (Tracks 11–17), band 3, 'Aspects of Proof'.

Frame 1

A claim

The square of any odd integer is odd.

If n is an integer and n is odd, **then** n^2 is odd.



Frame 2

A direct proof (of the claim in Frame 1)

Any odd integer is of the form $2u+1$, where $u \in \mathbb{Z}$.

Since n is odd, we have $n = 2u+1$.

$$\begin{aligned} n^2 &= (2u+1)^2 \\ &= \boxed{} \\ &= 2\,\boxed{} + 1 \\ &= 2v+1, \text{ where } v \in \mathbb{Z}. \end{aligned}$$

I know:
n is odd.

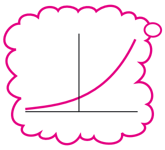
I want:
n^2 is odd.

So we have shown

if n is an odd integer, **then** n^2 is an odd integer.

Frame 3

Proving a function is one-one ..., or not!

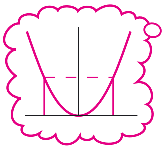


f is one-one

is equivalent to

for all a and b in the domain of f,
if $f(a) = f(b)$ then $a = b$.

a generality



g is <u>not</u> one-one

is equivalent to

there exist a and b in the domain of g with
$g(a) = g(b)$ and $a \neq b$.

a particular example

Frame 4

Activity 1.4 Prove f is one-one

Prove that the function $f: \mathbb{R} \to \mathbb{R}$ is one-one, where

$$f(x) = 2x - 1.$$

So we need to prove:
for all a and b in $\mathbb{R}$,
if $f(a) = f(b)$, then $a = b$.

Proof

Let a and b be arbitrary numbers in $\mathbb{R}$.

If $\quad f(a) = f(b)$

then $\quad \boxed{\qquad\qquad\qquad\qquad}$.

Hence $\quad \boxed{\qquad\qquad\qquad\qquad}$

and so $\quad a = b.$

Therefore f is one-one.

Frame 5

Activity 1.5 Prove g is not one-one

Prove that the function $g: \mathbb{R} \to \mathbb{R}$ is not one-one, where

$$g(x) = x^2 - 4x.$$

Proof

Let $\quad a = \boxed{\qquad\qquad}$

and $\quad b = \boxed{\qquad\qquad}$.

Then $\quad g(a) = \boxed{\qquad\qquad\qquad\qquad}$

and $\quad g(b) = \boxed{\qquad\qquad\qquad\qquad}$

and so $\quad g(a) = g(b).$

Hence g is not one-one.

You need
$a \neq b$

such that

$g(a) = g(b)$

There exists
such a pair
a and b

a counter-example

Frame 6

Arithmetic and geometric means

The arithmetic mean of two positive real numbers p and q is greater than or equal to their geometric mean. The means are equal only if $p = q$.

Proof

Arithmetic mean $\quad A = \dfrac{p+q}{2}$

Geometric mean $\quad G = \sqrt{pq}$

The definitions

$$\begin{aligned} A^2 - G^2 &= \left(\frac{p+q}{2}\right)^2 - (\sqrt{pq})^2 \\ &= \frac{p^2}{4} + \frac{q^2}{4} + \frac{2pq}{4} - pq \\ &= \frac{p^2}{4} + \frac{q^2}{4} - \frac{2pq}{4} = \frac{(p-q)^2}{4}. \end{aligned}$$

The algebraic expression for $A - G$ does not simplify but that for $A^2 - G^2$ does simplify.

Now $(p-q)^2 \geq 0$, with equality only if $p = q$.

So $A^2 = G^2$ if $p = q$, while if $p \neq q$, we have

$$A^2 - G^2 > 0.$$

That is $\quad A^2 > G^2$.

Now A and G are both positive.

Look back to their definitions

So $\quad A > G$.

Thus we have shown that $A \geq G$, with equality only if $p = q$.

Frame 7

A fallacious 'proof'

Can you see the error?

Proof

Let	$x = 1.$	(1)
Then we know that	$x^2 = x.$	(2)
Subtract 1 from each side	$x^2 - 1 = x - 1.$	(3)
Factorise the left-hand side	$(x+1)(x-1) = x - 1,$	(4)
and cancel $x - 1$	$x + 1 = 1.$	(5)
We deduce that	$x = 0.$	(6)
But we know that $x = 1$, so	$1 = 0.$	(7)

Frame 8

A proof by contradiction

Theorem

There are infinitely many prime numbers.

Proof

Start by assuming that the desired result is false.

Suppose that there are only finitely many prime numbers, say n of them. Then we can list them in a finite sequence:

$$p_1, p_2, p_3, \ldots, p_n,$$

where each prime appears exactly once.

Then use logical argument

Consider the number

$$m = p_1 \times p_2 \times p_3 \times \cdots \times p_n + 1.$$

This is bigger than any p_i, $i = 1, 2, \ldots, n$, and so does not appear in the list of primes. So m is not a prime, and hence must have a prime factor – and that must be in the list of primes. Suppose this factor is p_i.

to arrive at

Now if you divide m by p_i, you get a remainder of 1. This implies that p_i is not a factor of m.

a contradiction.

This is a contradiction, since we have

p_i is a factor of m, **and**
p_i is not a factor of m.

Hence our original assumption, that there are only finitely many prime numbers, must be false.

Conclusion: the desired result is true.

We conclude that there must be infinitely many prime numbers.

Frame 9

A question about primes

How far apart are the prime numbers? Is there a limit to the gap between one prime number and the next?

Imagine a list of all prime numbers less than 1000, ...

2, 3, 5, 7, 11, 13, ... , 883, 887, 907, 911, ... , 997,

1 2 2 4 2 4 20 4

... and the gaps.

Frame 10

A constructive proof

Theorem

For all $n \in \mathbb{N}$, there exists a set of n consecutive integers which are not prime.

We describe how to construct such a set.

Proof

Let n be any natural number.

Consider the set of n consecutive integers:

$$(n+1)!+2, (n+1)!+3, \ldots, (n+1)!+(n+1).$$

Check that
- there are n
- they are consecutive.

None of these is prime, since for any integer k between 2 and $n+1$, $(n+1)!$ is divisible by k, and so $(n+1)!+k$ is divisible by k.

So the theorem is proved; that is, for any natural number n there is a set of n consecutive integers that are not prime.

Test it – try a special case

If $n = 5$ then $n + 1 = 6$, so we have five consecutive numbers:

$6! + 2(= 722)$	$6! + 3(= 723)$	$6! + 4(= 724)$	$6!+ 5(= 725)$	$6! + 6(= 726)$
which is divisible by 2	which is divisible by 3	which is divisible by 4	which is divisible by 5	which is divisible by 6

Frame 11

Activity 1.6 A proof by exhaustion

Show that there are no solutions of the equation $x^2 = 2$ in $\mathbb{Z}_5$.

Proof

$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$.

$0^2 \equiv \boxed{} \pmod 5$

$1^2 \equiv \boxed{} \pmod 5$

$2^2 \equiv \boxed{} \pmod 5$

$3^2 \equiv \boxed{} \pmod 5$

$4^2 \equiv \boxed{} \pmod 5$

Check each case separately

So there are no solutions of $x^2 = 2$ in $\mathbb{Z}_5$.

Frame 12

Proof structures

Direct proof	Start with 'What I know'. Set up a chain of argument, ending with 'What I want'.
Contradiction	Include an assumption 'The required result is false'. Work to a contradiction.
Exhaustion	Check all possible cases individually.
Demonstration	Give an example to show that 'there exists'.
Counter-example	This establishes 'for all' to be false.

Frame 13

Proofs and you

Reading	Try some examples. Specialise Identify the type of proof structure. What is being proved? • Summarise.
Presenting	Is the structure of a recognisable type? Is each step justified, maybe: • algebraically correct; • from a definition; • from a known result?
Creating	Specialise/generalise • Why do the special cases work? • Does the reason generalise? I know/I want Work forward: • write down what you know; • write down definitions. Work back: • refine 'what you want' into simpler parts. Start with what you have and finish with what you want! Summarise what has been proved.

Frame 14

A result to be proved

> Let f and g be real increasing functions with domain $\mathbb{R}$, and let h be the sum of these two functions, that is,
> $$h(x) = f(x) + g(x) \qquad (x \in \mathbb{R}).$$
> Then h is also an increasing function.

Getting started

<u>I Know</u>

f is increasing

g is increasing

the definition of *h*

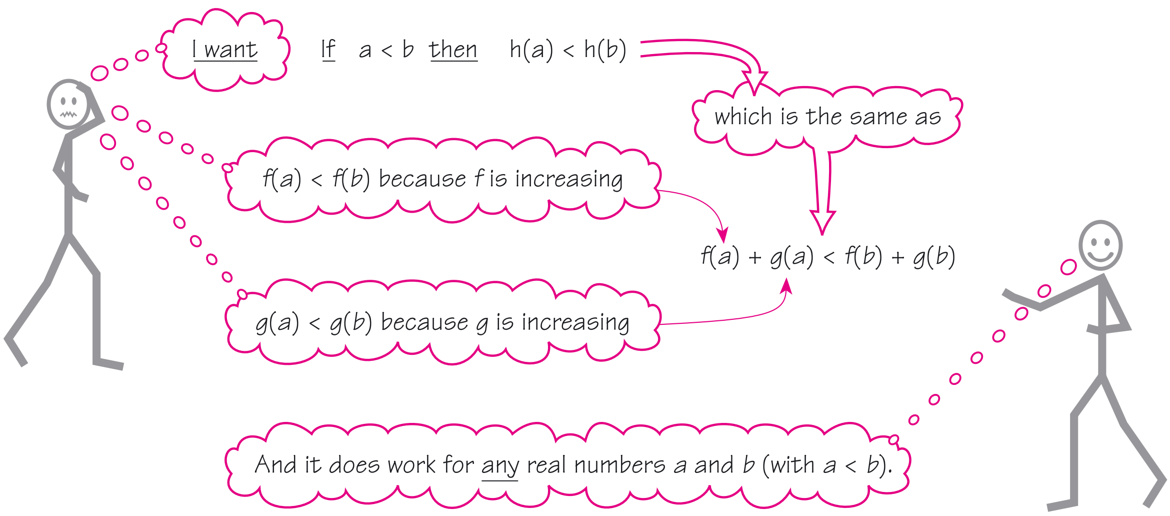
the definition of 'increasing'

<u>I Want</u>

h is increasing

Frame 15

Thinking



Frame 16

Tidy it up

Let a and b be arbitrary real numbers with $a < b$.

Then	$f(a) < f(b)$	(since f is increasing)	(1)
and	$g(a) < g(b)$	(since g is increasing).	(2)

Adding (1) and (2) gives

$$f(a) + g(a) < f(b) + g(b).$$

That is, $h(a) < h(b)$ (using the definition of h).

We have shown that: for all a and b in $\mathbb{R}$, if $a < b$ then $h(a) < h(b)$.

Hence h is increasing.

In the audio band, we discussed a number of ways in which a proof might be structured. In a result to be proved, certain statements are assumed to be true, often referred to as the **premises** of the proof, and the result asserts that the truth of some other statement, the **conclusion**, follows from these premises. The most natural form of proof starts with these premises, and works through a sequence of deductive steps, until we can deduce that the desired conclusion must be true. We referred to this as a **direct proof**.

We looked at various other forms of proof. To prove that some general statement is *not* true, it is sufficient to give a single example where the general statement does not hold (a **counter-example**).

A SINGLE EXAMPLE IS ENOUGH TO DISPROVE A GENERAL CLAIM

Where a statement asserts that something exists (as in Frame 10, for example), it is enough to exhibit an example satisfying the required conditions. Where a statement asserts that something holds true in a (small) finite number of cases, we can check each case individually (as in Frame 11). This approach is referred to as **proof by exhaustion**.

Proof by contradiction is an approach that may be useful in a variety of situations. Here, we assume that the desired conclusion is false. We then make deductions, based on the given premises together with this assumption, until we reach a contradiction; that is, we deduce the apparent truth of some statement that we know in fact to be false. We then conclude from this that our assumption that 'the desired conclusion is false' must be false, and hence that the desired conclusion is in fact true.

We might think of a direct proof as in Figure 1.1(a), and a proof by contradiction as in Figure 1.1(b).

Recall that 'argument' is an alternative word for 'proof'.

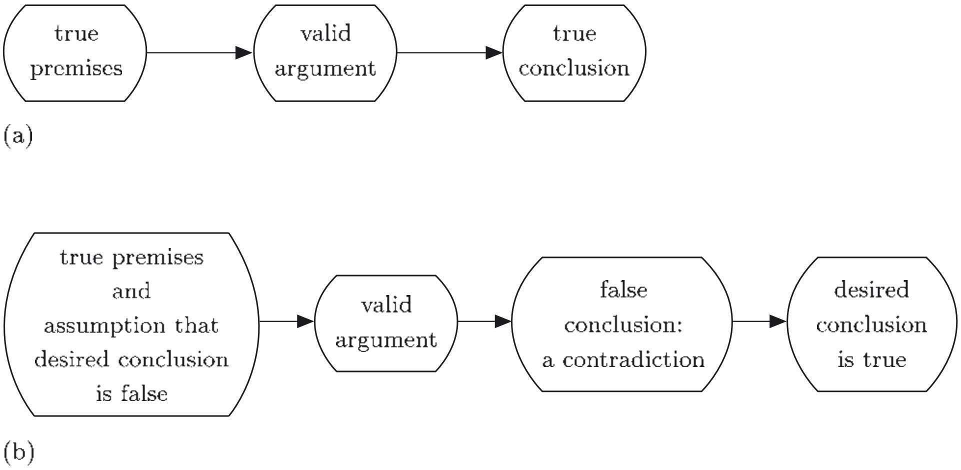


Figure 1.1 (a) Proving directly (b) Proving by contradiction

We aim to give valid arguments at all times! However, we do need to watch out for invalid ones. In general, if an argument leads to a conclusion that we know to be false, then either one of the premises on which the argument is based is false, or the argument itself is invalid. For example, if we start from true premises and end with a false conclusion, as we did in Frame 7, then the argument itself *must* be invalid.

An *invalid*, or *fallacious*, argument is one that cannot be justified.

The audio band concentrated on reading and presenting proofs. In Activity 1.2, we invited you to think about a conjecture, and how it might be justified. We ask you now to have a go at presenting a proof.

Activity 1.7 Presenting your own proof

On the basis of looking at particular cases, make a conjecture about the Two Sums Problem. Then try to give a proof of your conjecture.

A solution is given on page 43.

Summary of Section 1

We considered a number of examples of proofs, and identified some particular proof structures, summarised in Frame 12. We also noted some points to bear in mind when reading a proof in text, and when presenting or creating your own proofs. These are summarised in Frame 13.

Exercises for Section 1

Exercise 1.1

(a) Prove that the square of any even number is even.

(b) Prove that the product of two rational numbers is rational.

(c) Consider three consecutive natural numbers, say n, $n+1$ and $n+2$. Prove that one of these numbers must be divisible by 3. Use proof by exhaustion, considering each possible remainder of n on division by 3.

Exercise 1.2

Consider the statements in parts (a), (b) and (c) below, about a natural number n, where $n \geq 3$. Which do you think are true, and which false? In each case, either give a proof that the statement is true, or a counter-example proving that it is false.

(a) For $n \geq 3$, the number $n^3 - n$ is always even.

(b) For $n \geq 3$, the number $n^3 - n$ is always divisible by 3.

(c) For $n \geq 3$, the number $n^3 - n$ is always divisible by 4.

Exercise 1.3

(a) Prove that the function $f(x) = 1/x^2$ $(x > 0)$ is one-one.

(b) Prove that the function $g(x) = 1/x^2$ $(x \neq 0)$ is not one-one.

2 Implication and deduction

In Section 1 you saw some examples of proof in mathematics. We now focus on two important elements used in building proofs.

- ◇ **Propositions**, which are statements – in particular, a proposition must be either true or false.
- ◇ **Deduction**, which is the process of starting with some propositions that are known to be true and deducing that others are true.

Another word for 'statement' is 'assertion'.

As we do this, we shall introduce some notations which are often used in the succinct expression of results.

We look first at propositions. In mathematics, we are usually concerned with propositions that make statements about mathematical objects, such as:

the number 13 769 213 is prime;
there are no solutions of the equation $x^2 = 2$ in $\mathbb{Z}_5$;

or

the sum of two increasing real functions is increasing.

However, propositions may involve any statement that is either true or false, and they are not confined to those about mathematical objects. Statements such as:

my car has no petrol,

or

all the two-year-old cars for sale in this garage are red,

are also propositions.

We take it for granted when we formulate such propositions that we know who is being referred to by the 'my', or which garage is referred to by 'this'.

It is helpful to analyse the structure of propositions to some extent. You saw in Section 1 that certain phrases occur rather often in the statements that appear in mathematical proofs. Examples are:

'if ... then ...'; 'for all ...'; 'there exists ...'.

Here, we shall look at the first of these, which is particularly important in proofs. The other two are also important, but they are more complicated to deal with, and we shall not discuss them in detail. Examples of the 'if ... then ...' construction occur explicitly in statements such as:

if the integer n is odd, then n^2 is odd;
if the real function f is increasing, then f is one-one.

This construction is often implicit in other statements. For example, the proposition

the sum of two increasing real functions is increasing

asserts the same thing as

for all real functions f and g, if f and g are both increasing, then $f + g$ is increasing.

In writing text, it is often neater to express this proposition in the more compact form given first. However, to follow the structure of an argument involving this proposition, it may be helpful to recognise the way it is built up, which is made clearer in the second form of the proposition.

2.1 Combining propositions

To understand proofs, it is useful to be able to recognise when a complicated proposition is built up from simpler elements. Here we look at some of the fundamental elements used in building propositions.

Propositions may involve any statements that are either true or false, and we shall start by looking at some that do not refer to mathematical objects. For example, consider proposition (A) below.

If my car has no petrol, then it will not start. (A)

This proposition combines two elements using 'if ... then ...'. Proposition (B) below combines the same elements, but in a different way.

My car has no petrol and will not start. (B)

Activity 2.1 Same and different

Briefly consider the two propositions (A) and (B) above. Are they the same? If not, how do they differ?

A solution is given in the following text.

In fact, these two combinations are different. For (B) to be true it is required that my car will not start *and* that it has no petrol. On the other hand, proposition (A) asserts something about what will happen in the circumstance that my car has no petrol, but nothing about whether or not my car really does have no petrol. For (A) to be true, my car may or may not have petrol.

Incidentally, it is important to separate the process of formulating a proposition from the idea of it being true. We can formulate propositions that are true, or propositions that are false. Often, we formulate propositions that must be either true or false, but we do not know which is the case. Simply to formulate a proposition is not the same as claiming that it is true!

However, if we can recognise that a proposition is built up in some particular way by combining other propositions, then the truth or falsity of this combination follows inevitably from the truth or falsity of its elements. For example, proposition (B) above consists of the two elements

my car has no petrol,
my car will not start,

combined by using the word 'and'. Proposition (B) is true if each of these elements is true, and under no other circumstance.

We shall now – in typical mathematical fashion – introduce some symbols. Let a and b be the following propositions:

a means: my car has no petrol;

b means: my car will not start.

We denote the combination of these propositions in (B) above by

$a \wedge b$.

This is read as 'a and b'. We can combine any pair of propositions using the symbol $\wedge$, which is called **and**.

For example, if c and d are the propositions

c means: Henry is a dog,

d means: Henry has four legs,

then

$c \wedge d$ means: Henry is a dog and Henry has four legs.

We refer to any proposition formed by combining others as a **compound proposition**. The truth of the compound proposition $p \wedge q$ follows from knowledge of the truth of the two constituent propositions, p and q. We can show the rule for this in a table, as below. Such a table is referred to as a **truth table**.

p	q	$p \wedge q$
true	true	true
true	false	false
false	true	false
false	false	false

We refer to this table as 'the truth table for $\wedge$'.

Notice how the table shows the truth or falsity of the compound proposition $p \wedge q$ for each possible combination of truth or falsity of the two constituent propositions p and q. In particular, it shows that $p \wedge q$ is true only when each of p and q is true. The truth or falsity of any proposition is referred to as its **truth value**.

Now proposition (A) combines the two elementary propositions that we have denoted by a and b in a way different from that in proposition (B). We denote the combination of these propositions in (A) by

$a \Rightarrow b$.

This symbolism is read as 'if a then b', or as 'a implies b'. The compound proposition $a \Rightarrow b$ is called an **implication**. Again, we can use $\Rightarrow$ to combine other propositions. With c and d as defined above, we have:

$c \Rightarrow d$ means: if Henry is a dog, then Henry has four legs;

while

$d \Rightarrow c$ means: if Henry has four legs, then Henry is a dog.

Remember that to formulate a proposition is *not* the same as claiming that it is true.

Notice that the implications $c \Rightarrow d$ and $d \Rightarrow c$ have *different* meanings.

The truth table for $\Rightarrow$ is given below.

p	q	$p \Rightarrow q$	
true	true	true	(1)
true	false	false	(2)
false	true	true	(3)
false	false	true	(4)

The numbers in the table are just for reference.

The reason why some of these truth values for $p \Rightarrow q$ are appropriate is quite subtle. You may prefer just to regard the table above as the definition of what is meant by the symbol $\Rightarrow$, and not worry about how the entries in the table are arrived at. Since $p \Rightarrow q$ is read as 'if p then q', the entries in rows (1) and (2) are as you would expect. You may be surprised by the entries in rows (3) and (4), however, and feel that these entries should be 'false'. Notice though that if 'false' is entered in rows (3) and (4), then we obtain a truth table that is identical to that of $p \wedge q$. Since $p \Rightarrow q$ and $p \wedge q$ are different propositions, their truth tables need to be different.

For $p \wedge q$ to be true, p itself must be true, but for $p \Rightarrow q$ to be true, p need not be true.

A convenient way to express the content of a row in a truth table is to extend the use of the symbols $\wedge$ and $\Rightarrow$ to truth values themselves. For example, we may represent the content of row (3) of the truth table for $\Rightarrow$ as

'false $\Rightarrow$ true' is true,

and the last row of the truth table for $\wedge$ as

'false $\wedge$ false' is false.

Try using this form of expression in the next activity.

Activity 2.2 Deciding truth values for compound propositions

Suppose that propositions c and d have the meanings below.

c means: Henry is a dog.
d means: Henry has four legs.

Suppose also that, in fact, Henry is a cat (with four legs).

(a) State the truth value of each of the propositions c and d.

(b) Then, using the truth tables for $\wedge$ and $\Rightarrow$, determine the truth values of each of $c \wedge d$ and $c \Rightarrow d$.

Comment

(a) From the given information, c is false and d is true.

(b) With the truth values for c and d in part (a), we see that $c \wedge d$ is

false $\wedge$ true.

Thus $c \wedge d$ is false, from the third row of the truth table for $\wedge$.

Also, $c \Rightarrow d$ is

false $\Rightarrow$ true,

which, from row (3) of the truth table for $\Rightarrow$, is true.

Another word often used in combining propositions is 'or'. In English, this word may be used with two different meanings, either to mean 'a or b but not both a and b', or to mean 'either a or b or both a and b'. In any formalisation, we need to be clear as to which of these meanings is intended. For propositions p and q, we write $p \vee q$, where the symbol $\vee$ is called **or**, to mean the second of these, that is, 'either p or q or both p and q'. This meaning is sometimes called the 'inclusive' or.

Activity 2.3 The truth table for 'or'

Note that in our truth tables for the combination of two propositions, the four combinations of the truth values for p and q are always presented in the same order.

Complete the truth table for $p \vee q$.

p	q	$p \vee q$
true	true	
true	false	
false	true	
false	false	

A solution is given on page 43.

Converses

The interpretations of the propositions $p \Rightarrow q$ and $q \Rightarrow p$ are different. For example, with the propositions a and b given earlier:

$a \Rightarrow b$ means: if my car has no petrol, then it will not start;
$b \Rightarrow a$ means: if my car will not start, then it has no petrol.

These propositions do *not* mean the same thing. I would *expect* the first to be true, but not the second. If my car will not start, it might be through lack of petrol, but there are many other possible causes! Although these statements are related, they are different, and it is possible to confuse statements paired in this way if one is not careful. We refer to $q \Rightarrow p$ as the **converse** of $p \Rightarrow q$.

Similarly, $p \Rightarrow q$ is the converse of $q \Rightarrow p$.

Activity 2.4 Practice with propositions

The propositions c, d, e and f have the meanings given below.

c means: the number 123 456 is divisible by 3.
d means: the number 123 456 is divisible by 9.
e means: the sum of the digits in the number 123 456 is divisible by 3.
f means: the sum of the digits in the number 123 456 is divisible by 9.

(a) Express in English the meaning of each of the following propositions.

(i) $c \vee d$ (ii) $c \Rightarrow f$ (iii) $f \Rightarrow c$ (iv) $e \wedge d$

(b) Decide whether each of the propositions c, d, e and f is true or false. Then use the truth tables for combining propositions to determine the truth or falsity of each of the propositions in part (a)(i)–(iv).

Use results from Chapter D2, Section 2.

Solutions are given on page 43.

Variable propositions

In mathematics we are often concerned with statements involving variables. For example, we may wish to consider the 'propositions' $c(n)$ and $d(n)$ below, which involve an unspecified natural number n.

$c(n)$ means: n is divisible by 3.
$d(n)$ means: n is divisible by 9.

These are examples of **variable propositions**. A variable proposition can be considered to be a *condition* on the variable involved.

For each value of n in $\mathbb{N}$, we obtain specific propositions from $c(n)$ and $d(n)$; for example,

$c(123\,456)$ means: 123 456 is divisible by 3;
$d(44)$ means: 44 is divisible by 9.

Mathematical results often cover generalities. For example, it is true that if a natural number is divisible by 9, then it is divisible by 3. This result corresponds to the *truth* of the proposition

$$d(n) \Rightarrow c(n), \quad \text{for all } n \in \mathbb{N}.$$

Activity 2.5 Variable propositions

Let $c(n)$ and $d(n)$ be variable propositions with the meanings given above.

(a) Give examples of natural numbers n for which $d(n)$ is:

(i) true; (ii) false.

(b) Give examples of natural numbers n for which $c(n) \Rightarrow d(n)$ is:

(i) true; (ii) false.

Solutions are given on page 43.

Necessary and sufficient conditions

In mathematics, we are often interested in finding conditions that enable us to recognise that some property holds. For example, suppose we want to be able to recognise when a natural number is divisible by 18. To be divisible by 18, a number needs to be even. It also needs to be divisible by 3, and we can recognise this to be the case if the sum of the digits in the number is divisible by 3. We say that the conditions

See Chapter D2, Section 2.

n is even, and
the sum of the digits in n is divisible by 3,

are *necessary* conditions, in order that n is divisible by 18. However, even together, these conditions are not enough to enable us to deduce that a number n really *is* divisible by 18. For example, 24 satisfies both conditions, but it is not divisible by 18.

We refer to a condition, or collection of conditions, from which we *can* deduce a desired result, as *sufficient* conditions. For example, we can deduce that a natural number n is divisible by 3 from the condition

the natural number n is divisible by 9,

and we say this constitutes a sufficient condition that n is divisible by 3.

We can express these ideas of 'necessary' and 'sufficient' conditions using the language of propositions. Consider again the propositions $c(n)$ and $d(n)$, about an unspecified natural number n.

We sometimes omit 'variable' from 'variable proposition', when the context makes it evident that a proposition *is* variable, as with $c(n)$, $d(n)$ or $d(n) \Rightarrow c(n)$ here, for example.

$c(n)$ means: n is divisible by 3.
$d(n)$ means: n is divisible by 9.

The condition $d(n)$ is sufficient for the condition $c(n)$ to hold. This corresponds to the fact that

$$d(n) \Rightarrow c(n)$$

is true for any natural number n. Equivalently, we can say that the condition $c(n)$ is necessary in order that the condition $d(n)$ should hold.

In general, a proposition $p(x)$, about some variable x, is **sufficient** to ensure that $q(x)$ holds if

$$p(x) \Rightarrow q(x)$$

is true for all permitted values of x. It is **necessary** if the converse proposition

$$q(x) \Rightarrow p(x)$$

is true for all permitted values of x.

It is often of particular interest to find conditions that are *both* necessary *and* sufficient for some other condition to hold. The proposition $p(x)$ is both necessary and sufficient for $q(x)$ to hold if the combined proposition

$$(p(x) \Rightarrow q(x)) \wedge (q(x) \Rightarrow p(x))$$

Remember that $\wedge$ means 'and'.

is true for all permitted values of x. For any two propositions a and b, we write $a \Leftrightarrow b$ to mean the proposition $(a \Rightarrow b) \wedge (b \Rightarrow a)$. So $p(x)$ is necessary and sufficient for $q(x)$ to hold if

$$p(x) \Leftrightarrow q(x)$$

is true for all permitted values of x.

For example, a necessary and sufficient condition that a natural number n is divisible by 3 is that the sum of the digits in n is divisible by 3. Suppose that, for a natural number n,

See Chapter D2, Section 2.

$e(n)$ means: the sum of the digits in the number n is divisible by 3,

and $c(n)$ has the same meaning as above. Then each of the following propositions is true for all natural numbers n:

$$c(n) \Rightarrow e(n);$$

its converse

$$e(n) \Rightarrow c(n);$$

and these two propositions combined by 'and'

$$(c(n) \Rightarrow e(n)) \wedge (e(n) \Rightarrow c(n)),$$

which is the same as

$$c(n) \Leftrightarrow e(n).$$

The existence of a necessary and sufficient condition is often signalled by the words 'if and only if'. For example, we say 'the natural number n is divisible by 3 if and only if the sum of the digits in n is divisible by 3', and we read '$e(n) \Leftrightarrow c(n)$' as '$e(n)$ if and only if $c(n)$'.

Sometimes, mathematicians shorten 'if and only if' to 'iff'.

Activity 2.6 Necessary and sufficient conditions

For any natural number n, the propositions $a(n)$, $b(n)$, $c(n)$, $d(n)$, $e(n)$ and $f(n)$ have the meanings given below.

$a(n)$ means: the number n is even.
$b(n)$ means: the number n is divisible by 18.
$c(n)$ means: the number n is divisible by 3.
$d(n)$ means: the number n is divisible by 9.
$e(n)$ means: the sum of the digits in the number n is divisible by 3.
$f(n)$ means: the sum of the digits in the number n is divisible by 9.

See Chapter D2, Section 2.

(a) The sum of the digits of a natural number n is divisible by 9 if and only if n is divisible by 9. Express this fact as the truth of a compound variable proposition, formed from those above.

(b) Give a necessary and sufficient condition that a natural number n is divisible by 18. Express this condition in terms of the propositions above (other than $b(n)$).

(c) (i) Express the relationship between the propositions $b(n)$ and $c(n)$ in terms of 'necessary' and 'sufficient' conditions.

(ii) Give an example of a natural number n for which one of the two propositions in (i) is true and the other is false.

(iii) Which of the two propositions

$$b(n) \Rightarrow c(n) \quad \text{and} \quad c(n) \Rightarrow b(n)$$

is true for all $n \in \mathbb{N}$? Give an example of a number $n \in \mathbb{N}$ for which the other proposition is false.

Solutions are given on page 43.

2.2 Deduction

The use of 'if ... then ... ' statements forms a key element in the process of deduction. Suppose, for example, we know that both the following propositions are true.

If my car has no petrol, then it will not start.
My car has no petrol.

From these, we can deduce that

My car will not start.

To express this process of deduction in symbols, let a and b be the following propositions:

a means: my car has no petrol;
b means: my car will not start.

In the deduction above, we started with the knowledge that two propositions are true, these being $a \Rightarrow b$ and a, and we deduced from this knowledge that the proposition b must be true.

This form of deduction is the most typical single step in mathematical proofs. You establish the truth of two propositions p and $p \Rightarrow q$, and then deduce the truth of the proposition q. Understanding of this fundamental rule of logical reasoning dates back at least to the Stoic philosophers of ancient Greece. It is referred to by the Latin phrase *Modus Ponens*.

Or, more fully, *Modus ponendo ponens*, which might be translated as 'the way of establishing what lies behind'.

MODUS PONENS – POISED FOR ACTION

We can relate this rule to the truth table for $p \Rightarrow q$, given on page 21. Suppose we know that both of the propositions p and $p \Rightarrow q$ are true. Looking back to that truth table, notice that the truth of p means that we are in one of rows (1) or (2), while the truth of $p \Rightarrow q$ means that we are in one of rows (1), (3) or (4). The only row in which *both* these propositions are true is row (1), where q is also true. Thus if p and $p \Rightarrow q$ are both true, then we can indeed deduce that q is true.

Modus Ponens

From knowledge that the propositions

p and $p \Rightarrow q$

are both true, we can deduce that

q is true.

Modus Ponens is also known as the *Rule of Detachment*.

In mathematics, we quite often find that we can establish the truth of a general proposition of the form $p \Rightarrow q$. For example, the proposition

if the real function f is increasing, then f is one-one (2.1)

is true. The 'if ... then ...' here signals that this proposition is of the form $p \Rightarrow q$. Knowing (2.1) to be true, if we can show that some particular function f, such as $f(x) = x + 2$, *is* increasing, then we can *deduce* from the truth of (2.1) that f is one-one. This illustrates that some propositions are special cases of other more general propositions.

Activity 2.7 Practice with deductions

Below are five attempts at deductions. Which of these deductions are valid? (That is, in which cases does the truth of the conclusion follow from the truth of the given premises?) Where possible, define propositions p and q so that the form of the deduction is Modus Ponens.

(a) We know that:

> if the last bus has gone, then I must find a taxi;
> the last bus has gone.

We conclude that:

> I must find a taxi.

(b) We know that:

> all cats like fish;
> Winston is a cat.

We conclude that:

> Winston likes fish.

(c) We know that:

> if Spot is a dog, then Spot has four legs;
> Spot has four legs.

We conclude that:

> Spot is a dog.

(d) We know that:

> if my car has no petrol, then it will not start;
> my car has some petrol.

We conclude that:

> my car will start.

(e) We know that:

> increasing real functions are one-one;
> the function $f(x) = x^2$ $(x \in \mathbb{R})$ is not one-one.

We conclude that:

> the function $f(x) = x^2$ $(x \in \mathbb{R})$ is not increasing.

Comment

The deductions in parts (a), (b) and (e) are valid, but those in parts (c) and (d) are not.

(a) Define propositions p and q as follows:

> p means: the last bus has gone;
> q means: I must find a taxi.

Then the propositions that we know to be true are p and $p \Rightarrow q$, and the conclusion is that proposition q is true. So this argument is of the form Modus Ponens.

(b) This is a valid argument. We can regard it as a form of Modus Ponens if we rewrite the first premise as:

if x is a cat, then x likes fish.

The statement 'all cats like fish' asserts that this variable proposition is true, for all x in some unspecified set – say the set of all animals. Applying this variable proposition to the particular case where x is 'Winston' gives

if Winston is a cat, then Winston likes fish.

Now we can see the given deduction as an example of Modus Ponens, where

p means: Winston is a cat;
q means: Winston likes fish.

(c) Let p and q be the following propositions:

p means: Spot has four legs;
q means: Spot is a dog.

The propositions that we know to be true are $q \Rightarrow p$ and p. The argument attempts to deduce that q is true. This is not a valid argument. (Spot could be a cat!)

(d) Define the propositions p and q to be:

p means: my car has no petrol;
q means: my car will not start.

Then here we know that $p \Rightarrow q$ is true and that p is false. We attempt to deduce that q is false. Again, this is not a valid form of argument. (If my starter motor is broken, then my car will not start, even if it does have petrol!)

(e) This argument *is* valid. It is not of the form of Modus Ponens, though. We can relate it to Modus Ponens, but we need to combine that with the use of proof by contradiction, which was mentioned in Section 1.

Let f be the function $f(x) = x^2$ ($x \in \mathbb{R}$). Define the propositions p and q to be:

p means: f is increasing;
q means: f is one-one.

Then we know that the proposition $p \Rightarrow q$ is true, since it is a special case of the true general proposition (2.1) on page 27. Now *assume* that f is increasing; that is, that p is true. Then we can deduce that q is true; that is, that f is one-one. But we know this to be false and so have a contradiction. So our assumption that f is increasing must be false. Thus we conclude that f is not increasing.

Although the forms of argument in parts (c) and (d) of the Comment on Activity 2.7 are invalid, it is not uncommon to see attempts to use them both in mathematics and elsewhere.

Sequences of deductions

We now turn to a form of proof that is particularly useful for establishing results about natural numbers. We start with a simple example. Consider the sequence s_n generated by the recurrence system

$$s_1 = 0, \tag{2.2}$$

$$s_{n+1} = s_n \quad (n = 1, 2, 3, \ldots). \tag{2.3}$$

It seems clear that this recurrence ensures that $s_n = 0$ for all $n \in \mathbb{N}$.

We can deduce from equations (2.2) and (2.3) that

$$s_1 = 0 \quad \text{and} \quad \text{if } s_1 = 0, \text{ then } s_2 = 0.$$

We then deduce, by Modus Ponens, that $s_2 = 0$. In a similar way, we can deduce that $s_3 = 0$, $s_4 = 0$, and so on, so $s_n = 0$ for all $n \in \mathbb{N}$.

This proof has the form of a powerful general proof procedure, called *mathematical induction*. The logical structure of this method is as follows.

> **Mathematical induction**
> Let $p(n)$ be a variable proposition. If
> (a) $p(1)$ is true, and
> (b) the implication $p(k) \Rightarrow p(k+1)$ is true for all $k \in \mathbb{N}$,
> then
> $p(n)$ is true for all $n \in \mathbb{N}$.

We use the variable k to emphasise the difference between the statements

$p(k) \Rightarrow p(k+1)$ is true

and

$p(n)$ is true.

Mathematical induction enables us to deduce the truth of some variable proposition $p(n)$ for *all* natural numbers n. The power of mathematical induction lies in the fact that it is often easier to establish a result of the form '$p(k) \Rightarrow p(k+1)$ is true' than it is to establish the truth of $p(n)$ directly. The starting point is also crucial. For example, suppose that in our recurrence system example, equation (2.2) is different, say $s_1 = 2$. Then, if $p(n)$ means '$s_n = 0$', we can still deduce from equation (2.3) that

$$p(k) \Rightarrow p(k+1) \text{ is true for all } k \in \mathbb{N}.$$

We shall discuss in Section 3 how mathematical induction can be used in practice to prove many results about natural numbers.

But now $p(1)$ is *not* true, and so the chain of deductions cannot start.

This is just as well, since with $s_1 = 2$, we do not have $s_n = 0$ for all $n \in \mathbb{N}$.

Summary of Section 2

A proposition is a statement that must be either true or false. We can combine two propositions p and q in various ways:

- as $p \wedge q$, meaning 'p and q';
- as $p \vee q$, meaning 'either p or q or both p and q';
- as $p \Rightarrow q$, meaning 'if p then q';
- as $p \Leftrightarrow q$, meaning 'p if and only if q'.

We call $q \Rightarrow p$ the converse of $p \Rightarrow q$. Necessary conditions and sufficent conditions can be expressed using implications. A variable proposition has the form of a proposition whose truth or falsity depends on the value of some variable.

If we know that the propositions p and $p \Rightarrow q$ are both true, then we can deduce q to be true. This rule of reasoning is known as Modus Ponens. If we extend this method of deduction to an infinite sequence of such steps, then we obtain the method of proof by mathematical induction.

Exercises for Section 2

Exercise 2.1

(a) Suppose that the variable propositions $c(n)$ and $d(n)$, about an unspecified natural number n, have the meanings below.

$c(n)$ means: n is divisible by 7.
$d(n)$ means: n is divisible by 14.

For each of (i)–(iii), give an example of a natural number n for which:

(i) $d(n)$ is true and $c(n)$ is true;

(ii) $d(n)$ is false and $c(n)$ is true;

(iii) $d(n)$ is false and $c(n)$ is false.

Is there a value of n for which $d(n)$ is true and $c(n)$ is false?

(b) For each of the conditions given in (i)–(v), decide whether it is necessary, sufficient, necessary and sufficient, or neither necessary nor sufficient, in order that $x \equiv y \pmod{10}$.

(i) $x - y = 50$ (ii) $x - y = 5$ (iii) $x - y$ is divisible by 10

(iv) $x - y$ is divisible by 5 (v) $x - y$ is divisible both by 5 and by 2

(c) Let a and b be propositions for which a is true and b is false. What is the truth value of the proposition $a \Leftrightarrow b$?

Exercise 2.2

Below are three attempts at deductions. Which of these deductions are valid? Where possible, define propositions p and q so that the form of the deduction is Modus Ponens.

(a) We know that:

good high jumpers are tall;
Susie is a good high jumper.

We conclude that

Susie is tall.

(b) We know that:

everyone that I know is Norwegian;
Gunnar is Norwegian.

We conclude that

I know Gunnar.

(c) We know that:

poodles are white;
Rambo is not white.

We conclude that

Rambo is not a poodle.

3 Proof by mathematical induction

At the end of Section 2 we stated a method, proof by mathematical induction, which is valuable for establishing general statements that hold for all natural numbers. Here, you will see examples of how this method of proof is used, and you will practise its use. We start with an example.

This expression was required in Chapter C3, Activity 2.2, in order to calculate the Taylor polynomial approximation of degree n about 0 for f.

Suppose that we need an expression for the nth derivative (for $n \in \mathbb{N}$) of the function

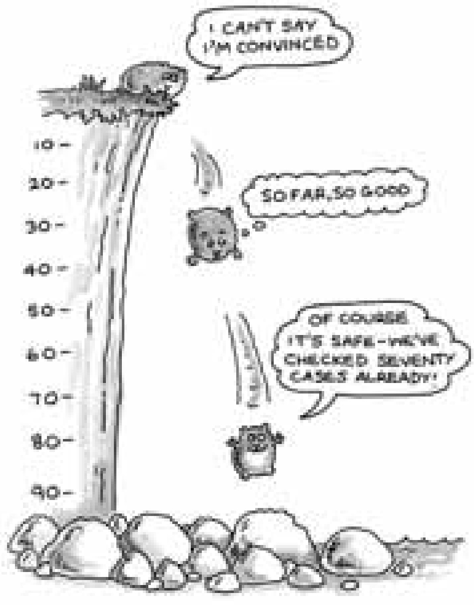
$$f(x) = \frac{1}{1-x} \quad (x < 1).$$

To find this, we can calculate the first few higher derivatives of f, and note that they show a pattern that can be generalised by the formula

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}, \qquad \text{for } n = 1, 2, 3, \ldots . \tag{3.1}$$

In non-mathematical contexts you may see the informal process of generalising from a pattern, as used here, called 'induction'. This should not be confused with 'mathematical induction', which is actually a process of *deduction.*

This approach of 'specialise/generalise' is a very useful way of discovering solutions to problems. However, a little caution is needed. The fact that the first few cases of some general statement hold true does not guarantee that the general statement itself is true. Equation (3.1) certainly *is* true for all natural numbers n, but it is desirable to have a method of proving this to be the case.



GENERALISATION REQUIRES PROOF

To prove that equation (3.1) does hold for all $n \in \mathbb{N}$, we can use mathematical induction. Such a proof requires that we establish two propositions.

(a) Equation (3.1) is true for $n = 1$.

Here $k = 1, 2, 3, \ldots$.

(b) If equation (3.1) is true for $n = k$, then it is also true for $n = k + 1$.

Suppose that $p(n)$ is the variable proposition

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}.$$

Then we need to show that
(a) $p(1)$ is true;
(b) $p(k) \Rightarrow p(k+1)$ is true for all $k \in \mathbb{N}$.

We can then use mathematical induction to deduce that $p(n)$ is true for all natural numbers n. In this way we establish the result *for all* n; that is, we prove that equation (3.1) holds *in general*.

Let us see how such a proof by mathematical induction might be laid out.

Example 3.1 Proof by mathematical induction

Prove equation (3.1) by mathematical induction.

Solution

(a) First we prove that $p(1)$ is true, namely that

$$f^{(1)}(x) = \frac{1!}{(1-x)^{1+1}} = \frac{1}{(1-x)^2}.$$

But $f^{(1)}(x)$ means $f'(x)$ and, by the Composite Rule,

$$f'(x) = \frac{-1}{(1-x)^2} \times (-1) = \frac{1}{(1-x)^2},$$

as required.

(b) We now show that, if equation (3.1) is true with $n = k$, then it is true with $n = k+1$. So, suppose that

$$f^{(k)}(x) = \frac{k!}{(1-x)^{k+1}}$$

is true. To find the $(k+1)$th derivative of f, we differentiate $f^{(k)}(x)$, which is given in the expression above. We obtain

$$f^{(k+1)}(x) = \frac{d}{dx}(f^{(k)}(x)) = \frac{d}{dx}\left(\frac{k!}{(1-x)^{k+1}}\right)$$
$$= \frac{-(k+1) \times k!}{(1-x)^{k+2}} \times (-1) = \frac{(k+1)!}{(1-x)^{k+2}}.$$

The expression on the right above is the same as that obtained by substituting $n = k+1$ in the expression on the right of equation (3.1); that is, we have shown that, *if* equation (3.1) holds with $n = k$, *then* it also holds with $n = k+1$, for $k = 1, 2, 3, \ldots$.

We conclude that equation (3.1) holds for all $n \in \mathbb{N}$, by mathematical induction.

Any such proof by mathematical induction contains two elements. We must establish the truth of the *inductive step*, that is, for all natural numbers k,

if $p(n)$ is true for $n = k$, then it is also true for $n = k+1$,

and the *initial step*, $p(1)$.

As a visualisation of the method of mathematical induction, consider an unending line of dominoes, set up so that:

> if any domino falls (to the right), then it will knock over the next domino (to the right).

If none of them is disturbed, then the dominoes will remain standing; see Figure 3.1(a). But suppose that the first domino is knocked down; see Figure 3.1(b). Then that will knock over the second; see Figure 3.1(c). The second will knock over the third, the third will knock over the fourth, and so on. Thus all the dominoes are knocked down.

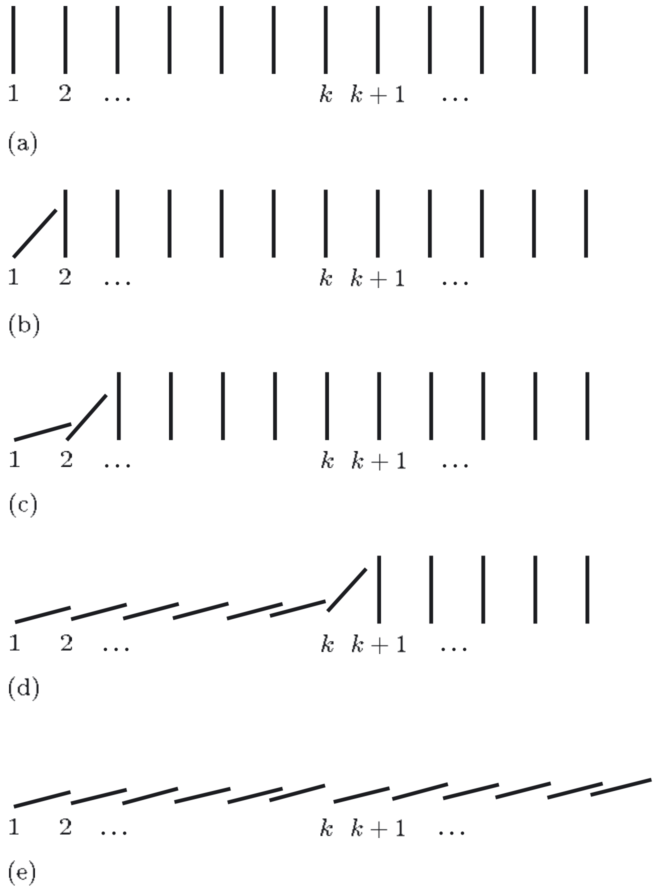


Figure 3.1 All the dominoes are knocked down

In Example 3.1, we used mathematical induction to confirm a formula arrived at by examining the first few expressions generated by an iterative process. In that case, the inductive step involved finding a derivative, but you have seen many different examples with a similar structure.

Activity 3.1 Proving formulas by mathematical induction

Give proofs by mathematical induction of each of the results below.

(a) Let the complex number z have polar form $\langle r, \theta \rangle$. Then, for all natural numbers n, the power z^n has polar form $\langle r^n, n\theta \rangle$. (Use the formula for multiplying two complex numbers in polar form.)

See Chapter D1, Section 3.

(b) For any natural number n, we have $10^n \equiv 1 \pmod 3$. (Use the fact that if $a \equiv b \pmod n$ and $c \equiv d \pmod n$, then $ac \equiv bd \pmod n$).

See Chapter D2, Theorem 1.2(e).

Solutions are given on page 44.

The next activity is included as a warning about the need for proofs to confirm the generalisation of patterns!

Activity 3.2 A pattern that does not generalise

Calculate the first five values given by the formula

$$u_n = n + \text{floor}(n/10) \quad (n = 1, 2, 3, \ldots).$$

Here floor(x) is equal to the greatest integer which is less than or equal to x.

Do these values show a pattern? Does the pattern hold for all $n \in \mathbb{N}$?

Comment

We obtain: $u_1 = 1$; $u_2 = 2$; $u_3 = 3$; $u_4 = 4$ and $u_5 = 5$. These values certainly do show a pattern, and this pattern suggests that $u_n = n$ for $n \in \mathbb{N}$. However, this pattern does not hold for *all* values of n; it breaks down as soon as n reaches 10. For example, $u_{10} = 11$ and $u_{105} = 115$.

Closed forms for recurrence systems

One context in which mathematical induction is often useful is to confirm closed forms for sequences defined by recurrence systems. Such a closed form may be suggested by examining the first few terms of the sequence.

Activity 3.3 Finding a pattern

Examine the first five terms generated by the recurrence system below. On the basis of these terms, suggest a closed form for s_n.

$$s_1 = 1,$$
$$s_{n+1} = s_n + 2n + 1 \qquad (n = 1, 2, 3, \ldots).$$

Comment

We obtain:

$$s_1 = 1;$$
$$s_2 = 1 + 2 + 1 = 4;$$
$$s_3 = 4 + 4 + 1 = 9;$$
$$s_4 = 9 + 6 + 1 = 16;$$
$$s_5 = 16 + 8 + 1 = 25.$$

These values are all squares, and suggest the formula

$$s_n = n^2, \qquad \text{for } n = 1, 2, 3, \ldots .$$

The values calculated in Activity 3.3 suggest a result, but do not prove it. To establish that the formula holds for *all* natural numbers, we need a suitable proof, and we can use mathematical induction. To prove a closed form for a recurrence sequence, we have to show that:

(a) the formula gives the correct value for the case $n = 1$;

(b) if the formula is correct for $n = k$, then it is also correct for $n = k + 1$ (where k may be any value in $\mathbb{N}$).

We can then conclude that the formula holds for all natural numbers n.

Activity 3.4 Checking a closed form

Prove, using mathematical induction, that if the sequence s_n is given by the recurrence system in Activity 3.3, then $s_n = n^2$ for $n = 1, 2, 3, \ldots$.

Comment

Let $p(n)$ be the variable proposition $s_n = n^2$. First, we confirm that $p(1)$ is true. With $n = 1$, we have $n^2 = 1^2 = 1$, and this *is* the value of s_1 specified in the recurrence system.

Next, we show that if $p(k)$ is true, that is, $s_k = k^2$, then $p(k+1)$ is true, that is, $s_{k+1} = (k+1)^2$. So, suppose that $s_k = k^2$. From the recurrence relation, we have

$$\begin{aligned} s_{k+1} &= s_k + 2k + 1 \\ &= k^2 + 2k + 1, \quad \text{since } p(k) \text{ is true,} \\ &= (k+1)^2, \end{aligned}$$

as required. So we have shown that if $p(k)$ is true, then $p(k+1)$ is true, for $k = 1, 2, 3, \ldots$.

Hence, by mathematical induction, we do have $s_n = n^2$ for all $n \in \mathbb{N}$.

The next activity uses mathematical induction to find the sum of

$$\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + \cdots + n^2.$$

Activity 3.5 Summing squares

Prove that

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \qquad \text{for } n = 1, 2, 3, \ldots.$$

Comment

Let $p(n)$ be the variable proposition

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1).$$

First we check that $p(1)$ is true. With $n = 1$ we have

$$\tfrac{1}{6}n(n+1)(2n+1) = \tfrac{1}{6}(1+1)(2+1) = 1.$$

On the other hand, with $n = 1$ we have

$$\sum_{r=1}^{n} r^2 = \sum_{r=1}^{1} r^2 = 1^2 = 1.$$

So $p(1)$ is true.

Next we check that if $p(k)$ is true, that is,

$$\sum_{r=1}^{k} r^2 = \tfrac{1}{6}k(k+1)(2k+1),$$

then $p(k+1)$ is true, that is,

$$\sum_{r=1}^{k+1} r^2 = \tfrac{1}{6}(k+1)(k+2)(2(k+1)+1).$$

We have

$$\begin{aligned}\sum_{r=1}^{k+1} r^2 &= (1^2 + 2^2 + \cdots + k^2) + (k+1)^2 \\ &= \sum_{r=1}^{k} r^2 + (k+1)^2 \\ &= \tfrac{1}{6}k(k+1)(2k+1) + (k+1)^2, \quad \text{since } p(k) \text{ is true,} \\ &= \tfrac{1}{6}(k+1)\left((2k^2+k) + 6(k+1)\right) \\ &= \tfrac{1}{6}(k+1)(2k^2+7k+6) \\ &= \tfrac{1}{6}(k+1)(k+2)(2k+3) \\ &= \tfrac{1}{6}(k+1)(k+2)(2(k+1)+1),\end{aligned}$$

as required. So we have shown that if $p(k)$ is true, then $p(k+1)$ is true, for $k = 1, 2, 3, \ldots$.

Thus, by mathematical induction, we have

$$\sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1), \quad \text{for } n = 1, 2, 3, \ldots,$$

as required.

The next activity looks at a similar result, for sums of cubes.

Activity 3.6 Summing cubes

Use proof by mathematical induction to show that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2, \quad \text{for } n = 1, 2, 3, \ldots .$$

A solution is given on page 44.

Proof by mathematical induction is useful in a variety of contexts. Recall that in Block C you met the Product Rule for differentiation,

$$(fg)' = f'g + fg'.$$

The next activity asks you to use this rule to prove a formula for the nth derivative of a certain function.

Activity 3.7 An nth derivative

Using mathematical induction, prove that, if f is the function

$$f(x) = xe^x,$$

then, for all n in $\mathbb{N}$, the nth derivative of f is given by the formula

$$f^{(n)}(x) = (n+x)e^x.$$

A solution is given on page 44.

Next we introduce a generalisation of the method of mathematical induction. Consider the following variable proposition, which is an inequality:

$$2^n > 4n.$$

This inequality is not true for all natural numbers n. For example, both the inequalities $2^1 > 4 \times 1$ and $2^2 > 4 \times 2$ are false. However, if we consider larger values of n, then the inequality does seem to hold, and with increasing ease. For example, we have

$$2^5 > 4 \times 5, \text{ since } 32 > 20 \quad \text{and} \quad 2^6 > 4 \times 6, \text{ since } 64 > 24.$$

Can we use mathematical induction to prove that this inequality holds, as long as n is sufficiently large?

We look first at the inductive step. Note that the inequality $2^n > 4n$ is equivalent to $2^n - 4n > 0$. So, suppose that the inequality holds with $n = k$; that is, $2^k > 4k$. Then

$$\begin{aligned} 2^{k+1} - 4(k+1) &= 2 \times 2^k - 4(k+1) \\ &> 2 \times 4k - 4(k+1), \quad \text{using } 2^k > 4k, \\ &= 8k - 4k - 4 = 4k - 4. \end{aligned}$$

This expression is greater than 0 for $k > 1$. So proof by mathematical induction works, except that we cannot start at $n = 1$.

But we can certainly start at $n = 5$, for example, since $2^5 > 4 \times 5$ is true. Since we have established that 'if the result holds for $n = k$, then it holds for $n = k + 1$', we can deduce that the result holds for $n = 6$, then for $n = 7$, and so on. Thus the result holds for all natural numbers n with $n \geq 5$.

In general, we can use *any* natural number, say N, for the initial step in mathematical induction.

Mathematical induction (generalised form)
Let N be a natural number and let $p(n)$ be a variable proposition. If
(a) $p(N)$ is true, and
(b) the implication $p(k) \Rightarrow p(k+1)$ is true for all $k \in \mathbb{N}$ with $k \geq N$,
then

$p(n)$ is true for all $n \in \mathbb{N}$ with $n \geq N$.

Again, we can visualise this method using dominoes. Suppose as before that each domino (at least from the Nth onwards) has been set up close enough to the next that we can be sure of the following.

If any domino falls (to the right), then it will knock over the next domino (to the right).

If none of them is disturbed, then the dominoes will remain standing; see Figure 3.2(a). But suppose that one domino is knocked down, this time the Nth; see Figure 3.2(b). Then that will knock over the $(N + 1)$th; see Figure 3.2(c). That will knock over the next, and so on.

Thus all the dominoes to the right of the Nth are knocked down.

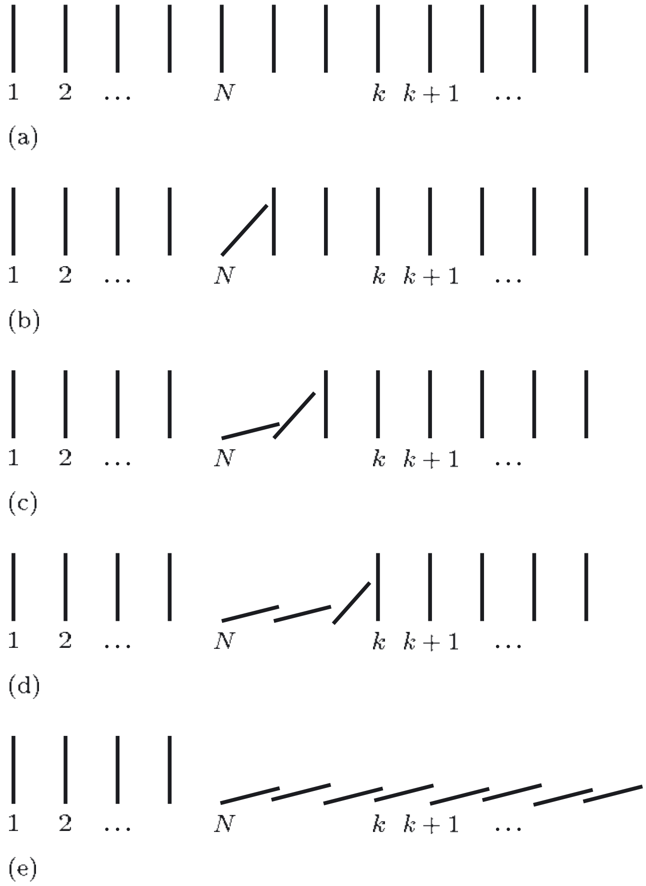


Figure 3.2 Dominoes from the Nth onwards are knocked down

Now we continue the discussion of the inequality $2^n > 4n$.

Activity 3.8 Finding a starting point

What is the smallest natural number n for which it is true that

$$2^n > 4n?$$

Comment

We have seen that the inequality holds for $n = 5$, and does not hold for $n = 2$. Checking the values $n = 3$ and $n = 4$, we obtain

$$2^3 > 4 \times 3 \text{ means } 8 > 12, \text{ which is false;}$$
$$2^4 > 4 \times 4 \text{ means } 16 > 16, \text{ which is also false.}$$

So $n = 5$ is the smallest value of $n \in \mathbb{N}$ for which the inequality holds.

Activity 3.9 Proving an inequality

This activity concerns the variable proposition $p(n)$, which means

$$3^n > 10n.$$

(a) What is the smallest value of $n \in \mathbb{N}$ for which $p(n)$ is true?

(b) Show that, for any natural number k, if $p(k)$ is true, then $p(k+1)$ is true.

(c) State the result that you can deduce from parts (a) and (b).

Solutions are given on page 44.

Activity 3.10 Another inequality

Let $p(n)$ be the variable proposition

$$2^n > n^2.$$

(a) Show that $p(1)$ is true.

(b) Show that

$$p(k) \Rightarrow p(k+1) \text{ is true for all } k \geq 3.$$

You may use the fact that

$$k^2 - 2k - 1 > 0, \text{ for } k \geq 3.$$

This inequality can be proved by completing the square:

$$k^2 - 2k - 1 = (k-1)^2 - 2.$$

(c) Can you conclude from your results in parts (a) and (b) that $p(n)$ is true for all $n \geq 3$?

(d) What is the smallest value of $N \in \mathbb{N}$ such that $p(n)$ is true for all $n \geq N$?

Comment

(a) $p(1)$ is the proposition $2^1 > 1^2$, that is, $2 > 1$, and so $p(1)$ is true.

(b) Suppose that $k \geq 3$, and that $p(k)$ is true; that is, $2^k > k^2$. Then

$$\begin{aligned} 2^{k+1} - (k+1)^2 &= 2 \times 2^k - (k^2 + 2k + 1) \\ &> 2k^2 - (k^2 + 2k + 1), \quad \text{since } p(k) \text{ is true,} \\ &= k^2 - 2k - 1 \\ &> 0, \text{ since } k \geq 3, \end{aligned}$$

by the given fact. Hence if $2^k > k^2$, then $2^{k+1} > (k+1)^2$; that is, $p(k) \Rightarrow p(k+1)$ is true for $k \geq 3$, as required.

(c) $p(3)$ is the proposition $2^3 > 3^2$, which is $8 > 9$, and this is false. Hence we certainly can *not* say that $p(n)$ is true for all natural numbers $n \geq 3$, since $p(n)$ is not true with $n = 3$.

(d) $p(4)$ is the proposition $2^4 > 4^2$, that is, $16 > 16$, which is false. The proposition $p(5)$ is $2^5 > 5^2$, that is, $32 > 25$, and this is true. We deduce, using mathematical induction, that $p(n)$ is true for all natural numbers $n \geq 5$. So the required value for N is 5.

Summary of Section 3

You have seen the use of proof by mathematical induction in various contexts. In particular, you used it to confirm results about sequences that are suggested by examining particular cases, and noting patterns. You saw that mathematical induction can be generalised to start at any natural number.

Exercises for Section 3

Exercise 3.1

Let s_n be the sequence defined by the recurrence system

$$s_1 = 1,$$
$$s_{n+1} = 2s_n + 1 \qquad (n = 1, 2, 3, \ldots).$$

(a) Calculate s_n for $n = 2, 3, 4$ and 5.

(b) Suggest a formula for s_n in terms of n, based on the initial term $s_1 = 1$ and the particular cases you found in part (a).

(c) Prove that your formula in part (b) holds for all $n \in \mathbb{N}$.

Exercise 3.2

Use mathematical induction to prove that

$$\sum_{r=1}^{n}(r \times r!) = (n+1)! - 1, \qquad \text{for } n = 1, 2, 3, \ldots.$$

Exercise 3.3

For $n \in \mathbb{N}$, consider the inequality below:

$$n! > 3^n.$$

(a) Prove that, for $k > 2$, if this inequality holds for $n = k$, then it holds for $n = k + 1$.

(b) Find the smallest value of n for which the inequality does hold.

(c) State the result that you can deduce from parts (a) and (b).

Summary of Chapter D4

The approach of 'specialise/generalise' provides a useful way of generating conjectures, but after making a conjecture, we need a proof to establish its truth (or falsity). We identified a number of strategies used in proofs, some that you have met before, and one important new technique, mathematical induction.

A proposition is a statement that must be either true or false. We identified some ways in which propositions can be combined, including 'and', 'or' and 'if ... then ...'. We introduced notation for these, and used this notation to give algebraic representations of two strategies used in proofs (Modus Ponens and mathematical induction).

Learning outcomes

You have been working towards the following learning outcomes.

Terms to know and use

Direct proof, counter-example, invalid argument, proof by contradiction, constructive proof, proof by exhaustion, premise, conclusion, proposition, deduction, compound proposition, truth table, truth value, implication, converse, variable proposition, necessary condition, sufficient condition, Modus Ponens, mathematical induction, initial step, inductive step.

Notation to know and use

$\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$.

Mathematical skills

- ◇ Use various proof techniques in suitable contexts: exhaustion; contradiction; counter-example; direct proof; demonstration (that 'there exists ...').
- ◇ Determine whether suitable propositions are true or false (including compound propositions). For a variable proposition, determine its truth or falsity for different values of the variable.
- ◇ Recognise suitable deductions of one step as valid or invalid, and, in appropriate cases, relate them to Modus Ponens.
- ◇ Give proofs by mathematical induction (including its generalised form) of results of the following types:

  formulas derived by iteration of some process;

  closed forms for first-order recurrence sequences;

  formulas for sums;

  inequalities involving natural numbers.

Solutions to Activities

Solution 1.7

There are infinitely many special cases you could have tried. They should have led you to the following conjecture.

Conjecture For any two real numbers x and y, with $x + y = 1$, we have

$$x^2 + y = x + y^2.$$

We can give a direct proof of this conjecture.

Proof Since $x + y = 1$, we have $y = 1 - x$. Then

$$x^2 + y = x^2 + (1 - x) = x^2 - x + 1,$$

while

$$\begin{aligned} x + y^2 = x + (1 - x)^2 &= x + x^2 - 2x + 1 \\ &= x^2 - x + 1. \end{aligned}$$

Hence $x^2 + y = x + y^2$, as required.

Notice that this proof is not dependent on which of x or y (if either) is larger. It holds for all real values of x and y, with $x + y = 1$.

Solution 2.3

The only combination leading to $p \vee q$ being false is when both of p and q are false, leading to the following truth table.

p	q	$p \vee q$
true	true	true
true	false	true
false	true	true
false	false	false

Solution 2.4

(a) (i) $c \vee d$ means 'the number 123 456 is divisible either by 3 or by 9, or by both'.

(ii) $c \Rightarrow f$ means 'if the number 123 456 is divisible by 3, then the sum of its digits is divisible by 9'.

(iii) $f \Rightarrow c$ means 'if the sum of the digits in 123 456 is divisible by 9, then the number 123 456 is divisible by 3'.

(iv) $e \wedge d$ means 'the sum of the digits in 123 456 is divisible by 3, and the number 123 456 is divisible by 9'.

(b) The sum of the digits in 123 456 is 21, which is divisible by 3 but not by 9. Thus e is true, while f is false. Then, using the digit sum tests for divisibility by 3 and 9 described in Chapter D2, Section 2, c is true, and d is false. Thus:

(i) $c \vee d$ is true $\vee$ false, which is true (see the second row of the truth table for $\vee$ given in Solution 2.3);

(ii) $c \Rightarrow f$ is true $\Rightarrow$ false, which is false (using the second row of the truth table for $\Rightarrow$);

(iii) $f \Rightarrow c$ is false $\Rightarrow$ true, which is true (using the third row of the truth table for $\Rightarrow$);

(iv) $e \wedge d$ is true $\wedge$ false, which is false.

Solution 2.5

Remember that $c(n)$ means that n is divisible by 3 and $d(n)$ means that n is divisible by 9.

(a) (i) If n is 18, say, then $d(n)$ is true.

(ii) If n is 15, say, then $d(n)$ is false.

(There are many other suitable examples.)

(b) (i) It follows from the truth table for $\Rightarrow$ that you can choose any value of n for which $c(n)$ is false, or for which $c(n)$ and $d(n)$ are both true. So, for example, $c(n) \Rightarrow d(n)$ is true with n equal to 14 or 18.

(ii) It follows from the truth table for $\Rightarrow$ that you need to choose a value of n for which $c(n)$ is true but $d(n)$ is false (so n is divisible by 3 but not by 9). For example, $c(n) \Rightarrow d(n)$ is false for n equal to 15.

Solution 2.6

(a) For all $n \in \mathbb{N}$,

$$f(n) \Leftrightarrow d(n) \text{ is true.}$$

(b) A natural number n is divisible by 18 if and only if n is divisible by both 2 and 9. Thus a necessary and sufficient condition that $b(n)$ be true is that

$$a(n) \wedge d(n) \text{ is true.}$$

(c) (i) If n is divisible by 18, then n is divisible by 3, but the converse is not true. So $c(n)$ is a necessary condition that $b(n)$ is true, and $b(n)$ is a sufficient condition that $c(n)$ is true.

(ii) If $n = 24$, for example, then $c(n)$ is true and $b(n)$ is false. (On the other hand, it is not possible to find a number $n \in \mathbb{N}$ for which $b(n)$ is true and $c(n)$ is false.)

(iii) The proposition $b(n) \Rightarrow c(n)$ is true for all n in $\mathbb{N}$, since any number n that is divisible by 18 is also divisible by 3. However, the proposition $c(n) \Rightarrow b(n)$ is false for certain values of n, such as $n = 24$.

Solution 3.1

(a) Let $p(n)$ be the variable proposition $z^n = \langle r^n, n\theta\rangle$. For $n = 1$, we have $z^1 = z$, which has polar form $\langle r, \theta\rangle = \langle r^1, 1 \times \theta\rangle$, as given by the formula with $n = 1$. Thus $p(1)$ is true.

Now suppose that $p(k)$ is true; that is, z^k has polar form $\langle r^k, k\theta\rangle$. Then $z^{k+1} = z \times z^k$, which has polar form

$$\begin{aligned}\langle r, \theta\rangle \times \langle r^k, k\theta\rangle &= \langle r \times r^k, \theta + k\theta\rangle, \\ &= \langle r^{k+1}, (k+1)\theta\rangle,\end{aligned}$$

using the rule for multiplication of complex numbers in polar form, so $p(k+1)$ is true. Thus $p(k) \Rightarrow p(k+1)$ is true for $k = 1, 2, 3, \ldots$.

Hence $p(n)$ is true for all $n \in \mathbb{N}$, by mathematical induction.

(b) Let $p(n)$ be the variable proposition $10^n \equiv 1 \pmod 3$. With $n = 1$ we have $10^1 = 10$, and

$$10 \equiv 1 \pmod 3$$

is true. Thus $p(1)$ is true.

Now suppose that $p(k)$ is true; that is,

$$10^k \equiv 1 \pmod 3.$$

We also know that $10^1 \equiv 1 \pmod 3$, and that we can multiply these two congruences (using Theorem 1.2(e) of Chapter D2) to obtain

$$10^1 \times 10^k \equiv 1 \times 1 \pmod 3;$$

that is, $10^{k+1} \equiv 1 \pmod 3$, so $p(k+1)$ is true. Thus $p(k) \Rightarrow p(k+1)$ is true for $k = 1, 2, 3, \ldots$.

We conclude, by mathematical induction, that $p(n)$ is true for all $n \in \mathbb{N}$.

Solution 3.6

Let $p(n)$ be the proposition $\displaystyle\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2$.

We have $\displaystyle\sum_{r=1}^{1} r^3 = 1^3 = 1$ and, for $n = 1$,

$$\tfrac{1}{4}n^2(n+1)^2 = \tfrac{1}{4} \times 1^2(1+1)^2 = 1,$$

so $p(1)$ is true.

Next, we show that if $p(k)$ is true, then $p(k+1)$ is true. By definition, we have

$$\begin{aligned}\sum_{r=1}^{k+1} r^3 &= (1^3 + 2^3 + 3^3 + \cdots + k^3) + (k+1)^3 \\ &= \sum_{r=1}^{k} r^3 + (k+1)^3 \\ &= \tfrac{1}{4}k^2(k+1)^2 + (k+1)^3, \text{ since } p(k) \text{ is true,} \\ &= \tfrac{1}{4}(k+1)^2\big(k^2 + 4(k+1)\big) \\ &= \tfrac{1}{4}(k+1)^2(k^2 + 4k + 4) \\ &= \tfrac{1}{4}(k+1)^2(k+2)^2,\end{aligned}$$

so $p(k+1)$ is true. Thus we have shown that if $p(k)$ is true, then $p(k+1)$ is true, for $k = 1, 2, 3, \ldots$.

We conclude, by mathematical induction, that $p(n)$ is true for all $n \in \mathbb{N}$.

Solution 3.7

Let $p(n)$ be the proposition $f^{(n)}(x) = (n+x)e^x$.

We first check that $p(1)$ is true. We can find the derivative of f using the Product Rule:

$$\begin{aligned}f^{(1)}(x) = f'(x) &= e^x + x\,e^x \\ &= (1+x)\,e^x.\end{aligned}$$

This agrees with the formula $(n+x)e^x$ given for $f^{(n)}(x)$, with $n = 1$, so $p(1)$ is true.

Now we show that if $p(k)$ is true, then $p(k+1)$ is true. We use the fact that $f^{(k+1)}(x)$ is the derivative of $f^{(k)}(x)$. Thus

$$\begin{aligned}f^{(k+1)}(x) &= \frac{d}{dx}\big(f^{(k)}(x)\big) \\ &= \frac{d}{dx}\big((k+x)e^x\big), \text{ since } p(k) \text{ is true,} \\ &= e^x + (k+x)e^x, \text{ by the Product Rule,} \\ &= (k+1+x)e^x,\end{aligned}$$

so $p(k+1)$ is true.

Thus we have shown that $p(1)$ is true and also that $p(k) \Rightarrow p(k+1)$ is true for $k = 1, 2, 3, \ldots$. We deduce, by mathematical induction, that $p(n)$ is true for all $n \in \mathbb{N}$.

Solution 3.9

(a) Each of $3^1 > 10$, $3^2 > 20$ and $3^3 > 30$ is false. The smallest value of n for which $p(n)$ is true is 4, since $3^4 > 4 \times 10$ is true.

(b) If $p(k)$ is true, then we have $3^k > 10k$. Now $p(k+1)$ is $3^{k+1} > 10(k+1)$, so we consider

$$\begin{aligned}3^{k+1} - 10(k+1) &= 3 \times 3^k - 10k - 10 \\ &> 3 \times 10k - 10k - 10, \\ &\qquad \text{since } p(k) \text{ is true,} \\ &= 20k - 10 \\ &> 0, \text{ since } k \geq 1.\end{aligned}$$

Thus $p(k) \Rightarrow p(k+1)$ is true for $k = 1, 2, 3, \ldots$.

(c) By mathematical induction, we conclude that the inequality $3^n > 10n$ holds for all natural numbers n with $n \geq 4$.

Solutions to Exercises

Solution 1.1

(a) An even number is of the form $2n$, where n is an integer. Its square is

$$(2n)^2 = 4n^2 = 2(2n^2).$$

Now $m = 2n^2$ is an integer, so the square $(2n)^2$ is of the form $2m$, where m is an integer, and so $(2n)^2$ is even.

(b) A rational number is of the form p/q, where p and q $(\neq 0)$ are integers. Consider the product of two such rational numbers. For integers p_1, q_1 $(\neq 0)$ and p_2, q_2 $(\neq 0)$, we have

$$\frac{p_1}{q_1} \times \frac{p_2}{q_2} = \frac{p_1 \times p_2}{q_1 \times q_2}.$$

The product of two integers is an integer, so $p_1 \times p_2$ and $q_1 \times q_2$ are integers. Also $q_1 \times q_2 \neq 0$. Hence the product of two rational numbers is rational.

(c) The number n must have a remainder on division by 3 that is 0, 1 or 2. We consider each of these three possibilities separately.

If $n \equiv 0 \pmod 3$, then n is divisible by 3.

If $n \equiv 1 \pmod 3$, then $n + 2 \equiv 3 \equiv 0 \pmod 3$. Hence $n + 2$ is divisible by 3.

If $n \equiv 2 \pmod 3$, then $n + 1 \equiv 3 \equiv 0 \pmod 3$. Hence $n + 1$ is divisible by 3.

In each case, one of the three numbers n, $n + 1$ or $n + 2$ is divisible by 3, as required.

Solution 1.2

In each case, you probably started by checking the conjecture in some special cases, perhaps trying n equal to each of 3, 4, 5 and 6.

(a) This statement is true. We can prove it as follows. We have

$$n^3 - n = n(n^2 - 1) = n(n-1)(n+1).$$

Since the natural numbers alternate between odd and even, we can be sure that one of the two numbers $n - 1$ or n is even. Any product involving an even number is even, so $(n-1)n(n+1)$ must be even, as required.

(b) This statement is true. As in part (a), we have $n^3 - n = (n-1)n(n+1)$. Now $n - 1$, n and $n + 1$ are three consecutive natural numbers. Thus, using the result of Exercise 1.1(c), one of these numbers must be divisible by 3. Hence the product $(n-1)n(n+1)$ must be divisible by 3, and so $n^3 - n$ is divisible by 3, as required.

(c) This statement is false. We give a single counter-example to show this. Take $n = 6$. Then $n^3 - n = 6^3 - 6 = 210$, which is *not* divisible by 4.

Solution 1.3

(a) To show that a function f is one-one, we consider any two points x_1 and x_2 in the domain of f, and show that

$$\text{if } f(x_1) = f(x_2), \text{ then } x_1 = x_2.$$

For the given f, suppose that $f(x_1) = f(x_2)$ for x_1 and x_2 in the domain of f; that is,

$$\frac{1}{x_1^2} = \frac{1}{x_2^2}, \text{ where } x_1 > 0 \text{ and } x_2 > 0.$$

Then $x_2^2 = x_1^2$, so $x_2 = x_1$ (since x_1 and x_2 are both positive), as required.

(b) To show that the function g is not one-one, we can give a single example of x_1 and x_2 in the domain of g, with $x_1 \neq x_2$, and $g(x_1) = g(x_2)$. For example, we have $2 \neq -2$, and

$$g(2) = \frac{1}{2^2} = \frac{1}{4},$$

while

$$g(-2) = \frac{1}{(-2)^2} = \frac{1}{4} = g(2).$$

Hence g is not one-one.

Solution 2.1

(a) There are many suitable examples in each case.

(i) We could take $n = 28$ (or any multiple of 14).

(ii) We could take $n = 21$ (or any multiple of 7 that is not a multiple of 14).

(iii) We could take $n = 10$ (or any number that is not a multiple of 7).

There is no value of n that is divisible by 14 but not divisible by 7.

(b) (i) This is a *sufficient* condition. If $x - y = 50$, then x is congruent to y modulo 10. But it is not a necessary condition. For example, $51 \equiv 31 \pmod{10}$, but $51 - 31$ is not equal to 50.

(ii) This condition is neither necessary nor sufficient.

(iii) This condition is both necessary and sufficient that $x \equiv y \pmod{10}$.

(iv) This is a necessary condition, but not sufficient. If $x \equiv y \pmod{10}$, then $x - y$ is divisible by 10, and so must be divisible by 5. However, this condition is not enough to ensure that $x \equiv y \pmod{10}$; for example, we might have $x - y = 15$, and x would then *not* be congruent to y modulo 10.

(v) This condition is equivalent to that in (iii), and is again a necessary and sufficient condition that $x \equiv y \pmod{10}$.

(c) We use the fact that

$$a \Leftrightarrow b \text{ means } (a \Rightarrow b) \wedge (b \Rightarrow a).$$

When a is true and b is false, we obtain

$$a \Rightarrow b \text{ is true} \Rightarrow \text{false, which is false,}$$

using the second row of the truth table for $\Rightarrow$, and

$$b \Rightarrow a \text{ is false} \Rightarrow \text{true, which is true,}$$

from the third row of that truth table. Then, since $a \Leftrightarrow b$ means $(a \Rightarrow b) \wedge (b \Rightarrow a)$,

$$a \Leftrightarrow b \text{ is false} \wedge \text{true, which is false,}$$

using the third row of the truth table for $\wedge$.

Solution 2.2

(a) This *is* a valid deduction. The form of deduction is, in essence, Modus Ponens. The first proposition can be re-expressed as

if a person is a good high jumper, then that person is tall,

which makes it clear that the proposition involves an implication. As given, the proposition asserts a generality. We are interested in a special case of that generality, applied to Susie. Substituting 'Susie' for the 'person' in the proposition above, we obtain

if Susie is a good high jumper, then Susie is tall.

If we now define propositions p and q with the meanings:

p means: Susie is a good high jumper,
q means: Susie is tall,

then the deduction has the form of Modus Ponens.

(b) This is *not* a valid deduction. The first proposition can be re-expressed as

if I know a person, then that person is Norwegian.

Replacing the general 'a person' by 'Gunnar', we have

if I know Gunnar, then Gunnar is Norwegian.

Thus we know the truth of two propositions of the form $p \Rightarrow q$ and q, and we are attempting to deduce that p is true. This is a fallacious form of argument.

(c) This *is* a valid deduction. It is not of the form given in Modus Ponens, though. It can be seen as a combination of proof by contradiction and Modus Ponens, as follows.

Assume that 'Rambo is a poodle' is true. From the first of the given propositions, we know that

if Rambo is a poodle, then Rambo is white,

is true. So we can deduce (using Modus Ponens) that 'Rambo is white' must be true. But this contradicts the second of the given propositions. Hence the proposition that 'Rambo is a poodle' must in fact be false. So we deduce that

Rambo is not a poodle.

Solution 3.1

(a) We are given that $s_1 = 1$. We find $s_2 = 3$; $s_3 = 7$; $s_4 = 15$; $s_5 = 31$.

(b) Successive powers of 2 are: 2, 4, 8, 16, 32. The values in part (a) are 1 less than these, suggesting the formula

$$s_n = 2^n - 1, \text{ for } n = 1, 2, 3, \ldots.$$

(c) Let $p(n)$ be the variable proposition $s_n = 2^n - 1$.

With $n = 1$, we have

$$2^n - 1 = 2^1 - 1 = 1 = s_1,$$

so $p(1)$ is true.

Now suppose that $p(k)$ is true; that is, $s_k = 2^k - 1$. Then, from the recurrence relation,

$$\begin{aligned} s_{k+1} &= 2s_k + 1 \\ &= 2(2^k - 1) + 1, \quad \text{since } p(k) \text{ is true,} \\ &= 2 \times 2^k - 2 + 1 \\ &= 2^{k+1} - 1, \end{aligned}$$

so $p(k+1)$ is true. Thus if $p(k)$ is true, then $p(k+1)$ is true, for $k = 1, 2, 3, \ldots$.

We deduce by mathematical induction that the formula $s_n = 2^n - 1$ is true for all $n \in \mathbb{N}$.

Solution 3.2

Let $p(n)$ be the variable proposition

$$\sum_{r=1}^{n} (r \times r!) = (n+1)! - 1.$$

For $n = 1$, the sum is

$$\sum_{r=1}^{1} (r \times r!) = 1 \times 1! = 1,$$

while the formula $(n+1)! - 1$ with $n = 1$ gives

$$2! - 1 = 2 - 1 = 1.$$

So $p(1)$ is true.

Next we assume that $p(k)$ is true, that is,

$$\sum_{r=1}^{k}(r \times r!) = (k+1)! - 1,$$

and deduce that $p(k+1)$ is true, that is,

$$\sum_{r=1}^{k+1}(r \times r!) = (k+2)! - 1.$$

We have

$$\begin{aligned}
\sum_{r=1}^{k+1}(r \times r!) &= \sum_{r=1}^{k}(r \times r!) + (k+1) \times (k+1)! \\
&= (k+1)! - 1 + (k+1) \times (k+1)!, \\
&\qquad\qquad \text{since } p(k) \text{ is true,} \\
&= (k+1)!(1+k+1) - 1 \\
&= (k+1)!(k+2) - 1 \\
&= (k+2)! - 1,
\end{aligned}$$

as required. Thus if $p(k)$ is true, then $p(k+1)$ is true, for $k = 1, 2, 3, \ldots$.

We deduce by mathematical induction that $p(n)$ is true for all $n \in \mathbb{N}$.

Solution 3.3

(a) Let $k > 2$, and suppose that $k! > 3^k$ is true. Then

$$\begin{aligned}
(k+1)! - 3^{k+1} &= (k+1)k! - 3 \times 3^k \\
&> (k+1)3^k - 3 \times 3^k, \\
&\qquad \text{since } k! > 3^k, \\
&= 3^k(k+1-3) \\
&= 3^k(k-2) \\
&> 0, \quad \text{since } k > 2.
\end{aligned}$$

Thus $(k+1)! > 3^{k+1}$ is also true. That is, if the inequality $n! > 3^n$ holds for $n = k$, then it holds for $n = k+1$, where $k > 2$.

(b) Looking at $n = 1, 2, \ldots 7$, we find the following values.

n	1	2	3	4	5	6	7
3^n	3	9	27	81	243	729	2187
$n!$	1	2	6	24	120	720	5040

So $n = 7$ is the smallest value of n for which $n! > 3^n$.

(c) Let $p(n)$ be the variable proposition $n! > 3^n$. We conclude from parts (a) and (b), by mathematical induction, that $p(n)$ is true for all natural numbers n with $n \geq 7$; that is,

$$n! > 3^n, \qquad \text{for } n = 7, 8, 9, \ldots .$$

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